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# Solutions of the quantum Yang–Baxter equation with extra non-additive parameters

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**Abstract.** We present a systematic technique to construct solutions to the Yang–Baxter equation which depend not only on a spectral parameter but in addition on further continuous parameters. These extra parameters enter the Yang–Baxter equation in a similar way to the spectral parameter but in a non-additive form.

We exploit the fact that quantum non-compact algebras such as  $U_q(su(1, 1))$  and type-I quantum superalgebras such as  $U_q(gl(1|1))$  and  $U_q(gl(2|1))$  are known to admit non-trivial one-parameter families of infinite-dimensional and finite-dimensional irreps, respectively, even for generic  $q$ . We develop a technique for constructing the corresponding spectral-dependent  $R$ -matrices. As examples, we work out the  $R$ -matrices for the three quantum algebras mentioned above in certain representations.

## 1. Introduction

Quantized universal enveloping algebras (quantum algebras) [1–3] provide a powerful tool for finding solutions to the spectral-dependent quantum Yang–Baxter equation (QYBE) [2, 4, 5]. There exists one such solution, with trigonometric dependence on the spectral parameter, for every pair of representations of any quantum affine Lie algebra. Through the work of many authors a large number of such solutions has now been constructed (see references in [6]). These solutions depend on a parameter  $q$ , the deformation parameter of the quantum algebras. This parameter  $q$  is very different in nature from the spectral parameter, because  $q$  does not enter into the Yang–Baxter equation.

It is clearly desirable to have families of solutions of the Yang–Baxter equation depending continuously on extra parameters, entering in a similar way to the spectral parameter. In this paper we develop a method for the construction of such families of solutions. The extra parameters enter the Yang–Baxter equation in a non-additive form. Special  $4 \times 4$  matrix solutions of this type have been constructed previously by solving the Yang–Baxter equation directly [7–10]. Our general group-theoretic method has the advantage of generating infinitely many solutions of any size. Also the chiral Potts model has a spectral parameter which enters the Yang–Baxter equation in a non-additive form. This solution arises from quantum groups at  $q$  a root of unity only [11, 12]. These are not related to our solutions however, in which the spectral parameter stays additive and the extra parameters are non-additive. Our solutions exist for generic  $q$ .

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Solutions of the Yang–Baxter equation have various different applications, for example as Boltzmann weights of integrable lattice models or as scattering matrices in integrable quantum field theories. In all these applications the freedom of having extra continuous parameters opens up new and exciting possibilities. We will come back to this in the discussion section.

The origin of the extra parameters in our solutions are the parameters which are carried by the irreps of the associated quantum algebra. Murakami [13] describes such parameters as colours carried by irreps. For the type-I quantum superalgebras, there are non-trivial one-parameter families of finite-dimensional irreps [14] for generic  $q$ . For quantum simple bosonic Lie algebras families of finite-dimensional representations are possible only when the deformation parameter  $q$  is a root of unity. However, they do admit parametrized families of unitary infinite-dimensional irreps even for generic  $q$ .

The main aim of this paper is to find solutions to the QYBE with extra continuous non-additive parameters associated with both the infinite-dimensional irreps of a quantum simple Lie algebra and the finite-dimensional irreps of quantum superalgebras. In section 2 we develop a systematic and useful technique which is very much in the spirit of the techniques in [6] designed to find solutions of the QYBE acting on the tensor product module of three different irreps of a quantum algebra. As concrete examples, we work out the solutions ( $R$ -matrices) associated with a one-parameter family of infinite-dimensional irreps for the quantum non-compact algebra  $U_q(su(1, 1))$  in section 3, and the  $R$ -matrices associated with a one-parameter family of finite-dimensional irreps for the quantum superalgebras  $U_q(gl(1|1))$  and  $U_q(gl(2|1))$  in section 4.

**2. General formalism**

Let  $G$  denote a simple Lie (super)algebra of rank  $r$  with generators  $\{e_i, f_i, h_i\}$  and let  $\alpha_i$  be its simple roots. Then the quantum (super)algebra  $U_q(G)$  can be defined with the structure of a ( $\mathbb{Z}_2$ -graded) quasi-triangular Hopf algebra [15, 4]. We will not give the full defining relations of  $U_q(G)$  here but mention that  $U_q(G)$  has a coproduct structure given by

$$\Delta(q^{h_i/2}) = q^{h_i/2} \otimes q^{h_i/2} \quad \Delta(a) = a \otimes q^{-h_i/2} + q^{h_i/2} \otimes a \quad (a = e_i, f_i). \tag{2.1}$$

The multiplication rule for the tensor product is defined for elements  $a, b, c, d \in U_q(G)$  by

$$(a \otimes b)(c \otimes d) = (-1)^{|b||c|}(ac \otimes bd) \tag{2.2}$$

where  $[a] \in \mathbb{Z}_2$  denotes the degree of the element  $a$ .

Let  $\pi_\Phi$  be a one-parameter family of irreps of  $U_q(G)$  afforded by the irreducible module  $V(\Phi)$  in such a way that the highest weight of the irrep depends on the parameter  $\Phi$ . Assume for any parameter  $\Phi$  that the irrep  $\pi_\Phi$  is affinizable, i.e. it can be extended to an irrep of the corresponding quantum affine (super)algebra  $U_q(\hat{G})$ . Consider an operator  $R^{\Phi_1\Phi_2}(x) \in \text{End}(V(\Phi_1) \otimes V(\Phi_2))$ , where  $x \in \mathbb{C}$  is the usual spectral parameter and  $\pi_{\Phi_1}, \pi_{\Phi_2}$  are two irreps from the one-parameter family. It has been shown by Jimbo [2] that a solution to the linear equations

$$\begin{aligned} R^{\Phi_1\Phi_2}(x)\Delta^{\Phi_1\Phi_2}(a) &= \bar{\Delta}^{\Phi_1\Phi_2}(a)R^{\Phi_1\Phi_2}(x) \quad \forall a \in U_q(G) \\ R^{\Phi_1\Phi_2}(x) & \left( x\pi_{\Phi_1}(e_0) \otimes \pi_{\Phi_2}(q^{-h_0/2}) + \pi_{\Phi_1}(q^{h_0/2}) \otimes \pi_{\Phi_2}(e_0) \right) \\ &= \left( x\pi_{\Phi_1}(e_0) \otimes \pi_{\Phi_2}(q^{h_0/2}) + \pi_{\Phi_1}(q^{-h_0/2}) \otimes \pi_{\Phi_2}(e_0) \right) R^{\Phi_1\Phi_2}(x) \end{aligned} \tag{2.3}$$

satisfies the QYBE in the tensor product module  $V(\Phi_1) \otimes V(\Phi_2) \otimes V(\Phi_3)$  of three irreps from the one-parameter family

$$R_{12}^{\Phi_1\Phi_2}(x)R_{13}^{\Phi_1\Phi_3}(xy)R_{23}^{\Phi_2\Phi_3}(y) = R_{23}^{\Phi_2\Phi_3}(y)R_{13}^{\Phi_1\Phi_3}(xy)R_{12}^{\Phi_1\Phi_2}(x). \tag{2.4}$$

In the above,  $\bar{\Delta} = T \cdot \Delta$ , with  $T$  the twist map defined by  $T(a \otimes b) = (-1)^{[\alpha][\beta]} b \otimes a$ ,  $\forall a, b \in U_q(G)$  and  $\Delta^{\Phi_1 \Phi_2}(a) = (\pi_{\Phi_1} \otimes \pi_{\Phi_2}) \Delta(a)$ ; also, if  $R^{\Phi_1 \Phi_2}(x) = \sum_i \pi_{\Phi_1}(a_i) \otimes \pi_{\Phi_2}(b_i)$ , then  $R_{12}^{\Phi_1 \Phi_2}(x) = \sum_i \pi_{\Phi_1}(a_i) \otimes \pi_{\Phi_2}(b_i) \otimes I$  etc. Jimbo also showed that the solution to (2.3) is unique, up to scalar functions. The multiplicative spectral parameter  $x$  can be transformed into an additive spectral parameter  $u$  by  $x = \exp(u)$ .

In all our equations we implicitly use the ‘graded’ multiplication rule of (2.2). Thus the  $R$ -matrix of a quantum superalgebra satisfies a ‘graded’ Yang–Baxter equation which, when written as an ordinary matrix equation, contains extra signs

$$\begin{aligned} & (R^{\Phi_1 \Phi_2}(x))_{\alpha\beta}^{\alpha'\beta'} (R^{\Phi_1 \Phi_3}(xy))_{\alpha'\gamma'}^{\alpha''\gamma''} (R^{\Phi_2 \Phi_3}(y))_{\beta'\gamma''}^{\beta''\gamma'''} (-1)^{[\alpha][\beta]+[\gamma][\alpha']+[y][\beta']} \\ &= (R^{\Phi_2 \Phi_3}(y))_{\beta'\gamma''}^{\beta''\gamma'''} (R^{\Phi_1 \Phi_3}(xy))_{\alpha'\gamma''}^{\alpha''\gamma'''} (R^{\Phi_1 \Phi_2}(x))_{\alpha'\beta'}^{\alpha''\beta''} (-1)^{[\beta][\gamma]+[\gamma][\alpha]+[\beta'][\alpha']} \end{aligned} \quad (2.5)$$

where  $[\alpha]$  denotes the degree of the basis vector  $v_\alpha$ . However, after a redefinition

$$(\bar{R}^{\Phi_1 \Phi_2})_{\alpha\beta}^{\alpha'\beta'} = (R^{\Phi_1 \Phi_2})_{\alpha\beta}^{\alpha'\beta'} (-1)^{[\alpha][\beta]} \quad (2.6)$$

the signs disappear from the equation. Thus any solution of the ‘graded’ Yang–Baxter equation arising from the  $R$ -matrix of a quantum superalgebra provides also a solution of the standard Yang–Baxter equation after the redefinition in (2.6).

Now we introduce the (graded) permutation operator  $P^{\Phi_1 \Phi_2}$  on the tensor product module  $V(\Phi_1) \otimes V(\Phi_2)$  such that

$$P^{\Phi_1 \Phi_2}(v_\alpha \otimes v_\beta) = (-1)^{[\alpha][\beta]} v_\beta \otimes v_\alpha \quad \forall v_\alpha \in V(\Phi_1) \quad v_\beta \in V(\Phi_2) \quad (2.7)$$

and set

$$\check{R}^{\Phi_1 \Phi_2}(x) = P^{\Phi_1 \Phi_2} R^{\Phi_1 \Phi_2}(x). \quad (2.8)$$

Then (2.3) can be rewritten as

$$\begin{aligned} \check{R}^{\Phi_1 \Phi_2}(x) \Delta^{\Phi_1 \Phi_2}(a) &= \Delta^{\Phi_2 \Phi_1}(a) \check{R}^{\Phi_1 \Phi_2}(x) \quad \forall a \in U_q(G) \\ \check{R}^{\Phi_1 \Phi_2}(x) (x \pi_{\Phi_1}(e_0) \otimes \pi_{\Phi_2}(q^{-h_0/2}) &+ \pi_{\Phi_1}(q^{h_0/2}) \otimes \pi_{\Phi_2}(e_0)) \\ &= (\pi_{\Phi_2}(e_0) \otimes \pi_{\Phi_1}(q^{-h_0/2}) + x \pi_{\Phi_2}(q^{h_0/2}) \otimes \pi_{\Phi_1}(e_0)) \check{R}^{\Phi_1 \Phi_2}(x) \end{aligned} \quad (2.9)$$

and in terms of  $\check{R}^{\Phi_1 \Phi_2}(x)$  the QYBE becomes

$$\begin{aligned} (I \otimes \check{R}^{\Phi_1 \Phi_2}(x)) (\check{R}^{\Phi_1 \Phi_3}(xy) \otimes I) (I \otimes \check{R}^{\Phi_2 \Phi_3}(y)) \\ = (\check{R}^{\Phi_2 \Phi_3}(y) \otimes I) (I \otimes \check{R}^{\Phi_1 \Phi_3}(xy)) (\check{R}^{\Phi_1 \Phi_2}(x) \otimes I) \end{aligned} \quad (2.10)$$

both sides of which act from  $V(\Phi_1) \otimes V(\Phi_2) \otimes V(\Phi_3)$  to  $V(\Phi_3) \otimes V(\Phi_2) \otimes V(\Phi_1)$ . Note that this equation, if written in matrix form, does not have extra signs in the superalgebra case. This is because the definition of the graded permutation operator in (2.7) includes the signs of (2.6).

In order to solve (2.9), we use a method similar to the one developed in [6] for a quite different problem. We will normalize the  $R$ -matrix  $\check{R}^{\Phi_1 \Phi_2}(x)$  in such a way that

$$\check{R}^{\Phi_1 \Phi_2}(x) \check{R}^{\Phi_2 \Phi_1}(x^{-1}) = I \quad (2.11)$$

which is usually called the unitarity condition in the literature. Consider three special cases:  $x = 0$ ,  $x = \infty$  and  $x = 1$ . For these special values of  $x$ ,  $\check{R}^{\Phi_1 \Phi_2}(x)$  satisfies the spectral-free, but extra non-additive parameter-dependent QYBE

$$(I \otimes \check{R}^{\Phi_1 \Phi_2})(\check{R}^{\Phi_1 \Phi_3} \otimes I) (I \otimes \check{R}^{\Phi_2 \Phi_3}) = (\check{R}^{\Phi_2 \Phi_3} \otimes I) (I \otimes \check{R}^{\Phi_1 \Phi_3})(\check{R}^{\Phi_1 \Phi_2} \otimes I). \quad (2.12)$$

Moreover, from (2.9), we have respectively, for  $x = 0$

$$\check{R}^{\Phi_1\Phi_2}(0)\Delta^{\Phi_1\Phi_2}(a) = \Delta^{\Phi_2\Phi_1}(a)\check{R}^{\Phi_1\Phi_2}(0) \quad \forall a \in U_q(G) \tag{2.13}$$

$$\check{R}^{\Phi_1\Phi_2}(0) (\pi_{\Phi_1}(q^{h_0/2}) \otimes \pi_{\Phi_2}(e_0)) = (\pi_{\Phi_2}(e_0) \otimes \pi_{\Phi_1}(q^{-h_0/2})) \check{R}^{\Phi_1\Phi_2}(0)$$

for  $x = \infty$

$$\check{R}^{\Phi_1\Phi_2}(\infty)\Delta^{\Phi_1\Phi_2}(a) = \Delta^{\Phi_2\Phi_1}(a)\check{R}^{\Phi_1\Phi_2}(\infty) \quad \forall a \in U_q(G) \tag{2.14}$$

$$\check{R}^{\Phi_1\Phi_2}(\infty) (\pi_{\Phi_1}(e_0) \otimes \pi_{\Phi_2}(q^{-h_0/2})) = (\pi_{\Phi_2}(q^{h_0/2}) \otimes \pi_{\Phi_1}(e_0)) \check{R}^{\Phi_1\Phi_2}(\infty)$$

and for  $x = 1$

$$\check{R}^{\Phi_1\Phi_2}(1)\Delta^{\Phi_1\Phi_2}(a) = \Delta^{\Phi_2\Phi_1}(a)\check{R}^{\Phi_1\Phi_2}(1) \quad \forall a \in U_q(G) \tag{2.15}$$

$$\check{R}^{\Phi_1\Phi_2}(1) (\pi_{\Phi_1}(e_0) \otimes \pi_{\Phi_2}(q^{-h_0/2}) + \pi_{\Phi_1}(q^{h_0/2}) \otimes \pi_{\Phi_2}(e_0))$$

$$= (\pi_{\Phi_2}(e_0) \otimes \pi_{\Phi_1}(q^{-h_0/2}) + \pi_{\Phi_2}(q^{h_0/2}) \otimes \pi_{\Phi_1}(e_0)) \check{R}^{\Phi_1\Phi_2}(1).$$

Equations (2.13), (2.14) and (2.15) respectively admit a unique solution for any given one-parameter family of irreps of  $U_q(G)$ , provided such representations are consistently affinizable. In the case of a multiplicity-free tensor product decomposition we may write

$$V(\Phi_1) \otimes V(\Phi_2) = \bigoplus_{\mu} V(\mu) \tag{2.16}$$

where  $\mu$  denotes a highest weight depending on the parameters  $\Phi_1$  and  $\Phi_2$ . Let  $\{|e_{\alpha}^{\mu}\rangle_{\Phi_1 \otimes \Phi_2}\}$  be an orthonormal basis for  $V(\mu)$  in  $V(\Phi_1) \otimes V(\Phi_2)$ .  $V(\mu)$  is also embedded in  $V(\Phi_2) \otimes V(\Phi_1)$  through the opposite coproduct  $\bar{\Delta}$ . Let  $\{|e_{\alpha}^{\mu}\rangle_{\Phi_2 \otimes \Phi_1}\}$  be the corresponding orthonormal basis†. Using these bases we define operators  $\mathcal{P}_{\mu}^{\Phi_1\Phi_2}$  and  $\mathbf{P}_{\mu}^{\Phi_1\Phi_2}$

$$\mathcal{P}_{\mu}^{\Phi_1\Phi_2} = \sum_{\alpha} |e_{\alpha}^{\mu}\rangle_{\Phi_1 \otimes \Phi_2} \langle e_{\alpha}^{\mu}|_{\Phi_1 \otimes \Phi_2} \quad \mathbf{P}_{\mu}^{\Phi_1\Phi_2} = \sum_{\alpha} |e_{\alpha}^{\mu}\rangle_{\Phi_2 \otimes \Phi_1} \langle e_{\alpha}^{\mu}|_{\Phi_2 \otimes \Phi_1} \tag{2.17}$$

Clearly the  $\mathcal{P}_{\mu}^{\Phi_1\Phi_2} : V(\Phi_1) \otimes V(\Phi_2) \rightarrow V(\mu) \subset V(\Phi_1) \otimes V(\Phi_2)$  are projection operators. The  $\mathbf{P}_{\mu}^{\Phi_1\Phi_2} : V(\Phi_1) \otimes V(\Phi_2) \rightarrow V(\mu) \subset V(\Phi_2) \otimes V(\Phi_1)$  are the elementary intertwiners, i.e.

$$\mathbf{P}_{\mu}^{\Phi_1\Phi_2} \Delta^{\Phi_1\Phi_2}(a) = \Delta^{\Phi_2\Phi_1}(a) \mathbf{P}_{\mu}^{\Phi_1\Phi_2} \quad \forall a \in U_q(G) \tag{2.18}$$

and  $\mathcal{P}_{\mu}^{\Phi_1\Phi_2}$  and  $\mathcal{P}_{\mu'}^{\Phi_1\Phi_2}$  satisfy the relations

$$\mathbf{P}_{\mu}^{\Phi_1\Phi_2} \mathcal{P}_{\mu'}^{\Phi_1\Phi_2} = \mathcal{P}_{\mu'}^{\Phi_2\Phi_1} \mathbf{P}_{\mu}^{\Phi_1\Phi_2} = \delta_{\mu\mu'} \mathbf{P}_{\mu}^{\Phi_1\Phi_2}$$

$$\mathbf{P}_{\mu}^{\Phi_2\Phi_1} \mathbf{P}_{\mu'}^{\Phi_1\Phi_2} = \delta_{\mu\mu'} \mathcal{P}_{\mu}^{\Phi_1\Phi_2} \tag{2.19}$$

$$\mathcal{P}_{\nu}^{\Phi_1\Phi_2} \mathcal{P}_{\nu'}^{\Phi_1\Phi_2} = \delta_{\nu\nu'} \mathcal{P}_{\nu}^{\Phi_1\Phi_2} \quad \sum_{\mu} \mathcal{P}_{\mu}^{\Phi_1\Phi_2} = I.$$

We can show [6] that the solutions of (2.13), (2.14) and (2.15) take the particularly simple forms

$$\check{R}^{\Phi_1\Phi_2}(0) = \sum_{\mu} \epsilon(\mu) q^{[C(\mu)-C(\Phi_1)-C(\Phi_2)]/2} \mathbf{P}_{\mu}^{\Phi_1\Phi_2}$$

$$\check{R}^{\Phi_1\Phi_2}(\infty) = \sum_{\mu} \epsilon(\mu) q^{-[C(\mu)-C(\Phi_1)-C(\Phi_2)]/2} \mathbf{P}_{\mu}^{\Phi_1\Phi_2} \tag{2.20}$$

$$\check{R}^{\Phi_1\Phi_2}(1) = \sum_{\mu} \mathbf{P}_{\mu}^{\Phi_1\Phi_2}$$

† For the precise definition of this basis see appendix C of [6].

where  $C(\Lambda) = (\Lambda, \Lambda + 2\rho)$  is the eigenvalue of the quadratic Casimir invariant of  $G$  in the irrep with highest weight  $\Lambda$ ,  $\rho$  is the half-sum of positive roots of  $U_q(G)$ , and  $\epsilon(\mu)$  is the parity of  $V(\mu)$  in  $V(\Phi_1) \otimes V(\Phi_2)$ .

The first two relations in (2.20) have been derived already by other methods in [3]. Here we illustrate the proof of the last relation in (2.20). From the unitarity condition (2.11) it follows that for  $x = 1$

$$\check{R}^{\Phi_1\Phi_2}(1)\check{R}^{\Phi_2\Phi_1}(1) = I. \quad (2.21)$$

We write  $\check{R}^{\Phi_1\Phi_2}(1)$  in the general form

$$\check{R}^{\Phi_1\Phi_2}(1) = \sum_{\mu} \rho_{\mu}(1) \mathbf{P}_{\mu}^{\Phi_1\Phi_2}. \quad (2.22)$$

Observing (2.21) we at once see that  $\rho_{\mu}(1)$  satisfies  $(\rho_{\mu}(1))^2 = 1$ , so that  $\rho_{\mu}(1) = \pm 1$ . By examining the limit  $\Phi_1 \rightarrow \Phi_2$ , which we can do because  $\Phi_1$  and  $\Phi_2$  are continuous parameters (such arguments are not valid for the case considered in [6] and thus the derivation of  $\check{R}(1)$  there is much more subtle), and using that, when  $\Phi_1 = \Phi_2$ ,  $\mathbf{P}_{\mu}^{\Phi_1\Phi_1}$  are the usual projection operators and  $\check{R}^{\Phi_1\Phi_1}(1)$  is the identity, one can conclude that the  $\rho_{\mu}(1)$  appearing in (2.22) must equal 1 identically, thus completing the proof.

We remark that in the present case  $\Phi_1$  etc are continuous parameters and so the parities  $\epsilon(\mu)$  in (2.20) can easily be worked out by examining the limit  $\Phi_1 \rightarrow \Phi_2$ , in contrast to the case considered in [6].

Multiplying the second equation in (2.13) by  $\mathcal{P}_{\mu}^{\Phi_2\Phi_1}$  from the left and by  $\mathcal{P}_{\nu}^{\Phi_1\Phi_2}$  from the right, and using (2.20) and (2.19) we obtain

$$\begin{aligned} \mathbf{P}_{\mu}^{\Phi_1\Phi_2} (\pi_{\Phi_1}(q^{h_0/2}) \otimes \pi_{\Phi_2}(e_0)) \mathcal{P}_{\mu}^{\Phi_1\Phi_2} &= \mathcal{P}_{\mu}^{\Phi_2\Phi_1} (\pi_{\Phi_2}(e_0) \otimes \pi_{\Phi_1}(q^{-h_0/2})) \mathbf{P}_{\mu}^{\Phi_1\Phi_2} \\ \epsilon(\mu) q^{C(\mu)/2} \mathbf{P}_{\mu}^{\Phi_1\Phi_2} (\pi_{\Phi_1}(q^{h_0/2}) \otimes \pi_{\Phi_2}(e_0)) \mathcal{P}_{\nu}^{\Phi_1\Phi_2} & \\ = \epsilon(\nu) q^{C(\nu)/2} \mathcal{P}_{\mu}^{\Phi_2\Phi_1} (\pi_{\Phi_2}(e_0) \otimes \pi_{\Phi_1}(q^{-h_0/2})) \mathbf{P}_{\nu}^{\Phi_1\Phi_2} & \quad \forall \mu \neq \nu. \end{aligned} \quad (2.23)$$

Similarly, from (2.14) and (2.15), we obtain

$$\begin{aligned} \mathbf{P}_{\mu}^{\Phi_1\Phi_2} (\pi_{\Phi_1}(e_0) \otimes \pi_{\Phi_2}(q^{-h_0/2})) \mathcal{P}_{\mu}^{\Phi_1\Phi_2} &= \mathcal{P}_{\mu}^{\Phi_2\Phi_1} (\pi_{\Phi_2}(q^{h_0/2}) \otimes \pi_{\Phi_1}(e_0)) \mathbf{P}_{\mu}^{\Phi_1\Phi_2} \\ \epsilon(\mu) q^{-C(\mu)/2} \mathbf{P}_{\mu}^{\Phi_1\Phi_2} (\pi_{\Phi_1}(e_0) \otimes \pi_{\Phi_2}(q^{-h_0/2})) \mathcal{P}_{\nu}^{\Phi_1\Phi_2} & \\ = \epsilon(\nu) q^{-C(\nu)/2} \mathcal{P}_{\mu}^{\Phi_2\Phi_1} (\pi_{\Phi_2}(q^{h_0/2}) \otimes \pi_{\Phi_1}(e_0)) \mathbf{P}_{\nu}^{\Phi_1\Phi_2} & \quad \forall \mu \neq \nu \end{aligned} \quad (2.24)$$

and

$$\begin{aligned} \mathbf{P}_{\mu}^{\Phi_1\Phi_2} (\pi_{\Phi_1}(e_0) \otimes \pi_{\Phi_2}(q^{-h_0/2})) \mathcal{P}_{\mu}^{\Phi_1\Phi_2} &= \mathcal{P}_{\mu}^{\Phi_2\Phi_1} (\pi_{\Phi_2}(e_0) \otimes \pi_{\Phi_1}(q^{-h_0/2})) \mathbf{P}_{\mu}^{\Phi_1\Phi_2} \\ \mathbf{P}_{\mu}^{\Phi_1\Phi_2} (\pi_{\Phi_1}(e_0) \otimes \pi_{\Phi_2}(q^{-h_0/2})) \mathcal{P}_{\nu}^{\Phi_1\Phi_2} & \\ = \mathcal{P}_{\mu}^{\Phi_2\Phi_1} (\pi_{\Phi_2}(e_0) \otimes \pi_{\Phi_1}(q^{-h_0/2})) \mathbf{P}_{\nu}^{\Phi_1\Phi_2} & \quad \forall \mu \neq \nu. \end{aligned} \quad (2.25)$$

In deriving (2.25), equations (2.23) and (2.24) have been considered.

Now the most general  $\check{R}^{\Phi_1\Phi_2}(x)$  satisfying the first equation in (2.9) may be written in the form

$$\check{R}^{\Phi_1\Phi_2}(x) = \sum_{V(\mu) \in V(\Phi_1) \otimes V(\Phi_2)} \rho_{\mu}(x) \mathbf{P}_{\mu}^{\Phi_1\Phi_2} \quad (2.26)$$

where  $\rho_{\mu}(x)$ , are unknown functions depending on  $x$ ,  $q$  and the extra non-additive parameters. Inserting the above equation into the second equation of (2.9) and multiplying

the resultant equation by  $\mathcal{P}_\mu^{\Phi_2\Phi_1}$  from the left and by  $\mathcal{P}_\nu^{\Phi_1\Phi_2}$  from the right, and then using (2.23), (2.24) and (2.25) to simplify the resulting equation, one finally finds that

$$\{\rho_\mu(x) (xq^{C(\mu)/2} + \epsilon(\mu)\epsilon(\nu)q^{C(\nu)/2}) - \rho_\nu(x) (q^{C(\mu)/2} + \epsilon(\mu)\epsilon(\nu)xq^{C(\nu)/2})\} \\ \times \mathcal{P}_\mu^{\Phi_1\Phi_2} (\pi_{\Phi_1}(e_0) \otimes \pi_{\Phi_2}(q^{-h_0/2})) \mathcal{P}_\nu^{\Phi_1\Phi_2} = 0 \quad \forall \mu \neq \nu. \tag{2.27}$$

In many cases it is possible to determine when  $\mathcal{P}_\mu^{\Phi_1\Phi_2} (\pi_{\Phi_1}(e_0) \otimes \pi_{\Phi_2}(q^{-h_0/2})) \mathcal{P}_\nu^{\Phi_1\Phi_2} \neq 0$  and thus to obtain a solution  $\rho_\mu(x)$  to the system of equations (2.27), recursively given by

$$\rho_\mu(x) = \rho_\nu(x) \frac{q^{C(\mu)/2} + \epsilon(\mu)\epsilon(\nu)xq^{C(\nu)/2}}{xq^{C(\mu)/2} + \epsilon(\mu)\epsilon(\nu)q^{C(\nu)/2}} \quad \forall \mu \neq \nu. \tag{2.28}$$

In the following sections we will work out three examples.

### 3. $U_q(su(1, 1))$ , its one-parameter family of irreps and $R$ -matrix with non-additive parameter

The quantum algebra  $U_q(su(1, 1))$  is defined by the generators  $\{e, f, q^h\}$  and relations

$$q^h e q^{-h} = q e \quad q^h f q^{-h} = q^{-1} f \quad [e, f] = -\frac{q^{2h} - q^{-2h}}{q - q^{-1}} \tag{3.1}$$

with the following coproduct:

$$\Delta(q^h) = q^h \otimes q^h \quad \Delta(e) = e \otimes q^{-h} + q^h \otimes e \quad \Delta(f) = f \otimes q^{-h} + q^h \otimes f. \tag{3.2}$$

Note here that we have used a slightly different normalization for the generator  $h$  compared with the one in section 2.

Infinite-dimensional unitary irreps of  $U_q(su(1, 1))$  have been described by several authors [16–18]. The classification is similar to that in the classical case, but there are some new features, notably the appearance of a ‘strange’ series of unitary irreps [18] with no classical analogue. For simplicity, we shall assume here that  $q \neq 1$  is real and positive, and consider the infinite-dimensional unitary irreps  $D^\pm(\Phi)$  from the so-called discrete series. The irrep  $D^-(\Phi)$  has a highest weight  $\Phi$  but no lowest weight, and  $D^+(\Phi)$  has a lowest weight  $-\Phi$  but no highest weight. In each case, the real parameter  $\Phi$  can take any negative value.

The structure of  $D^-(\Phi)$  is very simple: Let  $V_\Phi$  be a complex Hilbert space with orthonormal basis  $\{v_{\Phi+\mu}; \mu = 0, -1, -2, \dots\}$ , and set

$$\begin{aligned} h v_{\Phi+\mu} &= (\Phi + \mu) v_{\Phi+\mu} \\ e v_{\Phi+\mu} &= ([\mu]_q [\mu + 2\Phi + 1]_q)^{1/2} v_{\Phi+\mu+1} \\ f v_{\Phi+\mu} &= ([\mu - 1]_q [\mu + 2\Phi]_q)^{1/2} v_{\Phi+\mu-1}. \end{aligned} \tag{3.3}$$

where throughout this paper

$$[n]_q \equiv \frac{q^n - q^{-n}}{q - q^{-1}}. \tag{3.4}$$

The tensor product  $D^-(\Phi_1) \otimes D^-(\Phi_2)$  is completely reducible, and is easily seen to be

$$D^-(\Phi_1) \otimes D^-(\Phi_2) = \bigoplus_{\mu=0}^{-\infty} D^-(\Phi_1 + \Phi_2 + \mu). \tag{3.5}$$

The vector  $\Omega_{\Phi_1+\Phi_2+\mu}$ , corresponding to the highest weight  $\Phi_1+\Phi_2+\mu$ ,  $\mu = 0, -1, -2, \dots$  in the component  $D^-(\Phi_1 + \Phi_2 + \mu)$ , has the form

$$\Omega_{\Phi_1+\Phi_2+\mu} = c \sum_{i=0}^{\mu} (-1)^i q^{i(\Phi_1+\Phi_2+\mu+1)} \Gamma_q^{1/2}(2\Phi_1 + \mu - i + 1) \Gamma_q^{1/2}(\mu - i) \times \Gamma_q^{1/2}(2\Phi_2 + i + 1) \Gamma_q^{1/2}(i) v_{\Phi_1+\mu-i} \otimes v_{\Phi_2+i} \tag{3.6}$$

where  $c$  is a normalization constant, and  $\Gamma_q(z)$  is a  $q$ -gamma function [19] satisfying  $\Gamma_q(z + 1) = [z]_q \Gamma_q(z)$ . To verify (3.6), it is enough to check that  $\Omega_{\Phi_1+\Phi_2+\mu}$  has the correct weight and is annihilated by  $\Delta(e)$ .

Similar results hold for the structure of  $D^+(\Phi)$  and for the tensor product  $D^+(\Phi_1) \otimes D^+(\Phi_2)$ , but will not be used here.

All irreps of  $U_q(su(1, 1))$  are affinizable. In particular, we have the identification of  $e_0, h_0$  in section 2 with  $f, h$  in this section:  $e_0 \leftrightarrow f, h_0/2 \leftrightarrow -h$ . In the case of (3.5), equations (2.20) take the form

$$\begin{aligned} \check{R}^{\Phi_1\Phi_2}(0) &= \sum_{\mu=0}^{-\infty} (-1)^\mu q^{I(\Phi_1+\Phi_2+\mu)-I(\Phi_1)-I(\Phi_2)} \mathbf{P}_\mu^{\Phi_1\Phi_2} \\ \check{R}^{\Phi_1\Phi_2}(\infty) &= \sum_{\mu=0}^{-\infty} (-1)^\mu q^{-I(\Phi_1+\Phi_2+\mu)+I(\Phi_1)+I(\Phi_2)} \mathbf{P}_\mu^{\Phi_1\Phi_2} \\ \check{R}^{\Phi_1\Phi_2}(1) &= \sum_{\mu=0}^{-\infty} \mathbf{P}_\mu^{\Phi_1\Phi_2} \end{aligned} \tag{3.7}$$

where  $I(\Lambda) = \Lambda(\Lambda + 1)$  is the eigenvalue of the  $su(1, 1)$  quadratic Casimir invariant,  $I = h(h + 1) - fe = h(h - 1) - ef$ , in the irrep with highest weight  $\Lambda$ , and now  $\mathcal{P}_\mu^{\Phi_1\Phi_2} : D^-(\Phi_1) \otimes D^-(\Phi_2) \rightarrow D^-(\Phi_1 + \Phi_2 + \mu)$ . Equations (3.7) imply the identification of  $\epsilon(\mu), C(\mu)$  in (2.27):  $\epsilon(\mu) \leftrightarrow (-1)^\mu$  and  $C(\mu)/2 \leftrightarrow I(\Phi_1 + \Phi_2 + \mu)$ .

We will now determine  $\rho_\mu(x)$  in (2.27) for this case. Observe [5] that the tensor operator  $\Delta(q^{-h})(f \otimes q^h)$  behaves like a component of the adjoint tensor operator of  $U_q(su(1, 1))$ . Therefore,  $\mathcal{P}_\mu^{\Phi_1\Phi_2}(\pi_{\Phi_1}(f) \otimes \pi_{\Phi_2}(q^h)) \mathcal{P}_\nu^{\Phi_1\Phi_2}, \mu \neq \nu$ , vanishes unless the two highest weights  $\Phi_1 + \Phi_2 + \mu$  and  $\Phi_1 + \Phi_2 + \nu$ , associated with  $\mathcal{P}_\mu^{\Phi_1\Phi_2}$  and  $\mathcal{P}_\nu^{\Phi_1\Phi_2}$ , respectively, differ by a non-zero weight of the adjoint representation of  $U_q(su(1, 1))$ , that is by a root of  $su(1, 1)$ . This implies that  $\mathcal{P}_\mu^{\Phi_1\Phi_2}(\pi_{\Phi_1}(f) \otimes \pi_{\Phi_2}(q^h)) \mathcal{P}_\nu^{\Phi_1\Phi_2} \equiv 0$  for  $\mu \neq \nu$  unless  $\mu = \nu \pm 1$ . Therefore, from (2.27),

$$\frac{\rho_\mu(x)}{\rho_\nu(x)} = \frac{q^{I(\Phi_1+\Phi_2+\mu)} + (-1)^{\mu+\nu} x q^{I(\Phi_1+\Phi_2+\nu)}}{x q^{I(\Phi_1+\Phi_2+\mu)} + (-1)^{\mu+\nu} q^{I(\Phi_1+\Phi_2+\nu)}} \quad \mu = \nu \pm 1. \tag{3.8}$$

It follows immediately that

$$\rho_\mu(x) = \rho_0(x) \prod_{\nu=-1}^{\mu} \frac{1 - x q^{2(\Phi_1+\Phi_2+\nu)}}{x - q^{2(\Phi_1+\Phi_2+\nu)}} \quad \mu = -1, -2, \dots \tag{3.9}$$

which in turn leads to the quantum  $R$ -matrix

$$\check{R}^{\Phi_1\Phi_2}(x) = \sum_{\mu=0}^{-\infty} \prod_{\nu=-1}^{\mu} \frac{1 - x q^{2(\Phi_1+\Phi_2+\nu)}}{x - q^{2(\Phi_1+\Phi_2+\nu)}} \mathbf{P}_\mu^{\Phi_1\Phi_2} \tag{3.10}$$

(it should be understood that  $\prod_{\nu=-1}^0 (\dots) = 1$ ) where the scalar factor  $\rho_0(x)$  has been absorbed.



**4.  $U_q(gl(1|1)), U_q(gl(2|1))$ , one-parameter families of irreps and  $R$ -matrix with non-additive parameters**

It is well known [20] that type-I superalgebras admit non-trivial one-parameter families of finite-dimensional irreps which deform to provide one-parameter families of finite-dimensional irreps of the corresponding type-I quantum superalgebras [14]. Here we are only concerned with  $U_q(gl(m|n))$ , all irreps of which are known to be affinizable.

Choose  $\{\varepsilon_i\}_{i=1}^m \cup \{\bar{\varepsilon}_j\}_{j=1}^n$  as a basis for the dual of the Cartan subalgebra of  $gl(m|n)$  satisfying

$$(\varepsilon_i, \varepsilon_j) = \delta_{ij} \quad (\bar{\varepsilon}_i, \bar{\varepsilon}_j) = -\delta_{ij} \quad (\varepsilon_i, \bar{\varepsilon}_j) = 0. \tag{4.1}$$

Using this basis, any weight  $\Lambda$  may be written as

$$\Lambda \equiv (\Lambda_1, \dots, \Lambda_m | \bar{\Lambda}_1, \dots, \bar{\Lambda}_n) \equiv \sum_{i=1}^m \Lambda_i \varepsilon_i + \sum_{j=1}^n \bar{\Lambda}_j \bar{\varepsilon}_j \tag{4.2}$$

and the graded half-sum  $\rho$  of the positive roots of  $gl(m|n)$  is

$$2\rho = \sum_{i=1}^m (m - n - 2i + 1)\varepsilon_i + \sum_{j=1}^n (m + n - 2j + 1)\bar{\varepsilon}_j. \tag{4.3}$$

In what follows we will consider the one-parameter family of finite-dimensional irreducible  $U_q(gl(m|n))$ -modules  $V(\alpha)$  with highest weights of the form  $\Lambda(\alpha) = (0, \dots, 0|\alpha, \dots, \alpha)$ .

We first consider the one-parameter family of two-dimensional typical irreps  $V(\alpha)$  of  $U_q(gl(1|1))$  with the highest weight  $\Lambda(\alpha) = (0|\alpha)$ . Assuming that  $\alpha \neq 0 \neq \beta$ ,  $\alpha + \beta \neq 0$ , we have the following decomposition:

$$V(\alpha) \otimes V(\beta) = V(\Lambda_1) \oplus V(\Lambda_2) \tag{4.4}$$

where  $\Lambda_1 = (0|\alpha + \beta)$  and  $\Lambda_2 = (-1|\alpha + \beta + 1)$ . The Casimir operator takes the values

$$\begin{aligned} C(\alpha) &= -\alpha(\alpha + 1) \\ C(\beta) &= -\beta(\beta + 1) \\ C(\Lambda_1) &= -(\alpha + \beta)(\alpha + \beta + 1) \\ C(\Lambda_2) &= -(\alpha + \beta)^2 - 3(\alpha + \beta). \end{aligned} \tag{4.5}$$

By considering the limit  $q \rightarrow 1$ ,  $\alpha \rightarrow \beta$ , it follows that  $\epsilon(\Lambda_1) = -\epsilon(\Lambda_2) = 1$ . Also it is easy to conclude that  $\mathcal{P}_{\Lambda_1}^{\alpha\beta}(\pi_\alpha(e_0) \otimes \pi_\beta(q^{-h_0/2})) \mathcal{P}_{\Lambda_2}^{\alpha\beta} \neq 0$  and  $\mathcal{P}_{\Lambda_2}^{\alpha\beta}(\pi_\alpha(e_0) \otimes \pi_\beta(q^{-h_0/2})) \mathcal{P}_{\Lambda_1}^{\alpha\beta} \neq 0$ . Thus from (2.27) we have

$$\rho_{\Lambda_2}(x) = \frac{1 - xq^{\alpha+\beta}}{x - q^{\alpha+\beta}} \rho_{\Lambda_1}(x) \tag{4.6}$$

which gives rise to the properly normalized quantum  $R$ -matrix

$$\check{R}^{\alpha\beta}(x) = \mathbf{P}_{\Lambda_1}^{\alpha\beta} + \frac{1 - xq^{\alpha+\beta}}{x - q^{\alpha+\beta}} \mathbf{P}_{\Lambda_2}^{\alpha\beta} \tag{4.7}$$

where again the scalar factor  $\rho_{\Lambda_1}(x)$  has been absorbed.

It can be shown that the elementary intertwiners in the above equation take the form

$$\mathbf{P}_{\Lambda_1}^{\alpha\beta} = [\alpha + \beta]_q^{-1} \begin{pmatrix} [\alpha + \beta]_q & 0 & 0 & 0 \\ 0 & ([\alpha]_q [\beta]_q)^{1/2} q^{(\alpha+\beta)/2} & [\alpha]_q & 0 \\ 0 & [\beta]_q & ([\alpha]_q [\beta]_q)^{1/2} q^{-(\alpha+\beta)/2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$P_{\Lambda_2}^{\alpha\beta} = [\alpha + \beta]_q^{-1} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & ([\alpha]_q[\beta]_q)^{1/2} q^{-(\alpha+\beta)/2} & -[\beta]_q & 0 \\ 0 & -[\alpha]_q & ([\alpha]_q[\beta]_q)^{1/2} q^{(\alpha+\beta)/2} & 0 \\ 0 & 0 & 0 & [\alpha + \beta]_q \end{pmatrix}. \tag{4.8}$$

The details of derivations will be published in [21]. With the help of (4.8) the  $R$ -matrix (4.7) reads

$$\check{R}^{\alpha\beta}(x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -([\alpha]_q[\beta]_q)^{1/2} q^{(\alpha+\beta)/2} \cdot \frac{q-q^{-1}}{x-q^{\alpha+\beta}} & \frac{xq^\beta - q^\alpha}{x-q^{\alpha+\beta}} & 0 \\ 0 & \frac{xq^\alpha - q^\beta}{x-q^{\alpha+\beta}} & -([\alpha]_q[\beta]_q)^{1/2} q^{(\alpha+\beta)/2} \cdot \frac{x(q-q^{-1})}{x-q^{\alpha+\beta}} & 0 \\ 0 & 0 & 0 & \frac{1-xq^{\alpha+\beta}}{x-q^{\alpha+\beta}} \end{pmatrix}. \tag{4.9}$$

To avoid misunderstanding, we remind the reader that the  $R$ -matrix satisfies the QYBE acting on  $V(\alpha) \otimes V(\beta) \otimes V(\gamma)$

$$(I \otimes \check{R}^{\alpha\beta}(x))(\check{R}^{\alpha\gamma}(xy) \otimes I)(I \otimes \check{R}^{\beta\gamma}(y)) = (\check{R}^{\beta\gamma}(y) \otimes I)(I \otimes \check{R}^{\alpha\gamma}(xy))(\check{R}^{\alpha\beta}(x) \otimes I). \tag{4.10}$$

It should be pointed out that our six-vertex, free-fermion model  $R$ -matrix above is a trigonometric limit, up to a similarity transformation [22], of Bazhanov and Stroganov’s free-fermion, eight-vertex model  $R$ -matrix [7].

In the limit  $x = 0$ , we obtain the braid group representation  $\sigma^{\alpha\beta} \equiv \check{R}^{\alpha\beta}(0)$ :

$$\sigma^{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & ([\alpha]_q[\beta]_q)^{1/2} q^{-(\alpha+\beta)/2} (q - q^{-1}) & q^{-\beta} & 0 \\ 0 & q^{-\alpha} & 0 & 0 \\ 0 & 0 & 0 & -q^{-\alpha-\beta} \end{pmatrix}. \tag{4.11}$$

For the special case  $\alpha = \beta = 1$  this braid group representation is known. It was obtained from  $U_q(gl(1|1))$  in [23, 24].

We now come to the one-parameter family of four-dimensional typical irreps  $V(\alpha)$  [25] of  $U_q(gl(2|1))$  with the highest weight  $\Lambda(\alpha) = (0, 0|\alpha)$ . Assuming that  $\alpha \neq 0 \neq \beta$ ,  $\alpha + \beta \neq -1$ , we have the decomposition [26]

$$V(\alpha) \otimes V(\beta) = V(\Lambda_1) \oplus V(\Lambda_2) \oplus V(\Lambda_3) \tag{4.12}$$

where  $\Lambda_1 = (0, 0|\alpha + \beta)$ ,  $\Lambda_2 = (-1, -1|\alpha + \beta + 2)$  and  $\Lambda_3 = (0, -1|\alpha + \beta + 1)$ . Thus

$$\begin{aligned} C(\alpha) &= -\alpha(\alpha + 2) \\ C(\beta) &= -\beta(\beta + 2) \\ C(\Lambda_1) &= -(\alpha + \beta)(\alpha + \beta + 2) \\ C(\Lambda_2) &= 4 - (\alpha + \beta + 2)(\alpha + \beta + 4) \\ C(\Lambda_3) &= 3 - (\alpha + \beta + 1)(\alpha + \beta + 3). \end{aligned} \tag{4.13}$$

By considering the limit  $q \rightarrow 1$ ,  $\alpha \rightarrow \beta$ , we see that  $\epsilon(\Lambda_1) = \epsilon(\Lambda_2) = -\epsilon(\Lambda_3) = 1$ . Also we can conclude in this case that

$$\begin{aligned}
 \mathcal{P}_{\Lambda_1}^{\alpha\beta}(\pi_\alpha(e_0) \otimes \pi_\beta(q^{-h_0/2}))\mathcal{P}_{\Lambda_3}^{\alpha\beta} &\neq 0 \\
 \mathcal{P}_{\Lambda_3}^{\alpha\beta}(\pi_\alpha(e_0) \otimes \pi_\beta(q^{-h_0/2}))\mathcal{P}_{\Lambda_1}^{\alpha\beta} &\neq 0 \\
 \mathcal{P}_{\Lambda_3}^{\alpha\beta}(\pi_\alpha(e_0) \otimes \pi_\beta(q^{-h_0/2}))\mathcal{P}_{\Lambda_2}^{\alpha\beta} &\neq 0 \\
 \mathcal{P}_{\Lambda_2}^{\alpha\beta}(\pi_\alpha(e_0) \otimes \pi_\beta(q^{-h_0/2}))\mathcal{P}_{\Lambda_3}^{\alpha\beta} &\neq 0.
 \end{aligned}
 \tag{4.14}$$

Thus from (2.27) we obtain

$$\begin{aligned}
 \rho_{\Lambda_3}(x) &= \frac{1 - xq^{\alpha+\beta}}{x - q^{\alpha+\beta}} \rho_{\Lambda_1}(x) \\
 \rho_{\Lambda_2}(x) &= \frac{1 - xq^{\alpha+\beta}}{x - q^{\alpha+\beta}} \frac{1 - xq^{\alpha+\beta+2}}{x - q^{\alpha+\beta+2}} \rho_{\Lambda_1}(x)
 \end{aligned}
 \tag{4.15}$$

which lead to the quantum  $R$ -matrix

$$\check{R}^{\alpha\beta}(x) = \mathbf{P}_{\Lambda_1}^{\alpha\beta} + \frac{1 - xq^{\alpha+\beta}}{x - q^{\alpha+\beta}} \mathbf{P}_{\Lambda_3}^{\alpha\beta} + \frac{1 - xq^{\alpha+\beta}}{x - q^{\alpha+\beta}} \frac{1 - xq^{\alpha+\beta+2}}{x - q^{\alpha+\beta+2}} \mathbf{P}_{\Lambda_2}^{\alpha\beta}
 \tag{4.16}$$

where again the scalar factor  $\rho_{\Lambda_1}(x)$  has been absorbed. The explicit expressions for the elementary intertwiners  $\mathbf{P}_{\Lambda_i}^{\alpha\beta}$  appearing in this expression as  $16 \times 16$  matrices will be published in [21].

### 5. Conclusion

We have developed a systematic technique for constructing solutions to the QYBE with extra continuous non-additive parameters. The technique is a generalization of that used in [6] for a different problem. We have treated in particular solutions associated to families of representations of the quantum algebras  $U_q(su(1, 1))$ ,  $U_q(gl(1|1))$  and  $U_q(gl(2|1))$ . The technique can be applied to other quantum (super)algebras with families of irreducible highest-weight or lowest-weight representations. Recently we have obtained many new results for type-I quantum affine algebras in [27]. We can also apply the technique to families of representations of Yangians and will then obtain rational  $R$ -matrices with extra parameters. Our general expressions (2.20) for the braid generators can also be used to construct multivariable link invariants from quantum (super)algebras.

We expect that the possibility of extra continuous parameters in solutions of the Yang–Baxter equation will open up many new applications. The physical interpretation of the extra parameters will depend on the particular application.

To give an example, the scattering matrices for quantum excitations in integrable two-dimensional quantum field theories are given by solutions of the Yang–Baxter equation. The ratios between the masses of the quantum excitations are determined by the locations of the poles of the  $S$ -matrices. Because these poles are fixed in all known non-trivial crossing symmetric  $R$ -matrices, these  $R$ -matrices are able to describe only theories in which the ratios of the quantum masses are fixed to particular values. In our  $R$ -matrices the locations of the poles depend on the extra parameters. So if our  $R$ -matrices are used as  $S$ -matrices, these parameters have the interpretation of adjustable quantum masses in integrable theories. This application of the  $R$ -matrices derived in this paper is studied in [28].

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## References

- [1] Drinfeld V G 1986 *Proc. ICM, Berkeley* 1 798
- [2] Jimbo M 1985 *Lett. Math. Phys.* 10 63; 1986 *Lett. Math. Phys.* 11 247
- [3] Reshetikhin N 1987 Quantized universal enveloping algebras, the Yang–Baxter equation and invariants of links: I, II, *LOMI preprints* E-4-87, E-17-87
- [4] Bracken A J, Gould M D and Zhang R B 1990 *Mod. Phys. Lett. A* 5 831
- [5] Zhang R B, Gould M D and Bracken A J 1991 *Nucl. Phys. B* 354 625
- [6] Delius G W, Gould M D and Zhang Y-Z 1994 On the construction of trigonometric solutions of the Yang–Baxter equation *Universität Bielefeld and University of Queensland preprint* BI-TP/94/13, UQMATH-94-02, hep-th/9405030
- [7] Bazhanov V V and Stroganov Yu G 1985 *Theor. Math. Phys.* 62 253
- [8] Hlavatý L 1987 *J. Phys. A: Math. Gen.* 20 1661
- [9] Ge M-L and Xue K 1991 *J. Phys. A: Math. Gen.* 24 L895
- [10] Hlavatý L 1992 *J. Phys. A: Math. Gen.* 25 1395
- [11] Bazhanov V V, Kashaev R M, Mangazeev V V and Stroganov Yu 1990  $(\mathbb{Z}_n^\otimes)^{n-1}$  generalization of the chiral Potts model *Preprint*
- [12] Date E, Jimbo M, Miki K and Miwa T 1991 *Commun. Math. Phys.* 137 133
- [13] Murakami J 1993 *Pacific J. Math.* 157 109
- [14] Gould M D and Scheunert M 1993 Classification of finite dimensional unitary irreps for  $U_q(\mathfrak{gl}(m|n))$  *University of Queensland preprint*
- [15] Kulish P P and Reshetikhin N Y 1989 *Lett. Math. Phys.* 32 3261
- [16] Bernard D and LeClair A 1989 *Phys. Lett.* 227B 417
- [17] Kalnins E G, Manocha H L and Miller W 1992 *J. Math. Phys.* 33 2365
- [18] Burban I M and Klimyk A U 1993 *J. Phys. A: Math. Gen.* 26 2139
- [19] McAnally D S 1993  $q$ -exponential and  $q$ -gamma functions *University of Queensland preprint*
- [20] Kac V G 1977 *Adv. Math.* 26 8
- [21] Delius G W, Gould M D and Zhang Y-Z 1994 Two explicit solutions of the Yang–Baxter equation with extra continuous parameters *Universität Bielefeld and University of Queensland preprint* UQMATH-94-04 (in preparation)
- [22] Bracken A J, Delius G W, Gould M D and Zhang Y-Z 1993 Infinite families of gauge-equivalent  $R$ -matrices and gradations of quantized affine algebras *Universität Bielefeld and University of Queensland preprint* BI-TP/94/12, UQMATH-93-10, hep-th/9310183 (*J. Mod. Phys. B* in press)
- [23] Liao L and Song X C 1991 *Mod. Phys. Lett. A* 6 959
- [24] Rozansky L and Saleur H 1992 *Nucl. Phys. B* 376 461
- [25] Links J R and Gould M D 1992 *Lett. Math. Phys.* 26 187
- [26] Links J R and Zhang R B 1994 *J. Math. Phys.* 35 1377
- [27] Delius G W, Gould M D, Links J R and Zhang Y-Z 1994 On type-I quantum affine superalgebras *Universität Bielefeld and University of Queensland preprint* BI-TP/94/33, UQMATH-94-05, hep-th/9408006
- [28] Delius G W 1994 *Universität Bielefeld preprint* BI-TP/94/34 (in preparation)